Europ. J. Combinatorics (1986) **7**, 75–92

Quotients of Coxeter Complexes and Buildings with Linear Diagram

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1. INTRODUCTION

We investigate some combinatorial properties of a certain class of simplicial complexes which are associated with quotients of Coxeter groups. These simplicial complexes are subcomplexes of Coxeter complexes and buildings which were introduced and developed by Tits [16] [17]. The subcomplexes considered are called quotient Coxeter complexes and quotient buildings. Here we are concerned with those quotients that are order complexes of posets. We characterize such quotient Coxeter complexes and buildings as those that have linear Coxeter diagram. We also show that the poset (with a top and bottom element adjoined) is a lattice (for quotient Coxeter complexes and for non-quotient buildings) and is lexicographically shellable (for both quotient Coxeter complexes and quotient buildings). The shellability of posets is a concept that grew out of the theory of Cohen–Macaulay posets (cf. [1] [4] [10] [13]). Lexicographical shellability is a special type of poset shellability (cf. [2] [5] [6]).

The finite irreducible Coxeter groups have been classified by Coxeter [8]. The symmetry groups of the regular polytopes are the only finite irreducible Coxeter groups with linear Coxeter diagram. The Coxeter complex associated with the symmetry group of a regular polytope is the barycentric subdivision of the polytope. The barycentric subdivision is just the order complex of the lattice of faces of the polytope. In [6] it is shown that the lattice of faces of a polytope is lexicographically shellable. Hence all finite Coxeter complexes with connected linear Coxeter diagram are order complexes of lexicographically shellable lattices. This special case of our results is therefore easily derived from known results and is in part motivation for our study. Our proofs are intrinsic in that they do not make use of any classification schemes and they hold for infinite as well as finite complexes. Some of the results presented here are extensions of results of Tits [16], Surowski [15], and Vince [20] (see Section 6).

In [3] Björner shows that all Coxeter complexes and buildings are shellable and he presents a number of interesting combinatorial, algebraic, and topological consequences. In [11] Garsia and Stanton consider quotient Coxeter complexes and quotient buildings and illustrate their significance. They show that Björner's shellability results hold for quotients. These results of Björner and Garsia and Stanton are what motivated us to consider the lexicographical shellability of posets whose order complexes are quotients of Coxeter complexes and buildings.

I would like to express my gratitude to A. Björner for introducing me to his work on Coxeter complexes and buildings in an inspiring series of lectures given at U.C. San Diego and for his valuable suggestions. I also want to thank A. Garsia for his encouragement and for informing me of his work with Stanton on quotients.

In the next three sections we present preliminary definitions and results that will be needed in the remaining sections.

2. BASIC TERMINOLOGY

Let P be a poset. We say that P is *bounded* if there exists a top element $\hat{1}\in P$ and a bottom element $\hat{0}\in P$ such that $\hat{0}\leq x\leq\hat{1}$ for all $x\in P$. Given any poset P, let $\hat{P}$ denote

* Partially supported by NSF MCS81-03474.

0195-6698/86/010075+18 $02.00/0

the bounded poset obtained from P by adjoining a bottom element $\hat{0}$ and a top element $\hat{1}$. P is said to be *pure* if all maximal chains, $x_0 < x_1 < \cdots < x_r$ are finite and have the same length r. P is said to be *graded* if it is bounded and pure. If P is graded, then the *proper part* of F is defined to be the poset $P - \{\hat{0}, \hat{1}\}$ and is denoted by $\bar{P}$. Any element x of a pure poset has a well defined *rank* $\rho(x)$ equal to the common length of all unrefinable chains from a minimal element to x. By the *length* of a pure poset P we mean the rank of a maximal element. We say that y *covers* x in P and write $x \to y$ if $y > x$ and $x < z \leq y$ implies that $z = y$.

We will use the symbol $[n]$ to denote the set $\{1, 2, \ldots, n\}$. Let Δ be a simplicial complex. The maximal faces of Δ are called *chambers*. We say that Δ is *pure d-dimensional* if all chambers are of dimension d; i.e. they contain $d+1$ vertices. We say that Δ is *balanced* if Δ is pure d-dimensional and its vertex set V can be partitioned $V = \biguplus_{i \in [d+1]} V_i$ ($\biguplus$ denotes disjoint union) such that $|C \cap V_i| = 1$ for all chambers C and all $i \in [d+1]$. The partition is called a *balanced partition*. It is convenient to think of $[d+1]$ as a set of *colors*, the condition being that every chamber has exactly one vertex of each color.

To any poset P, we associate the simplicial complex $\Delta(P)$ of all chains of P called the *order complex* of P. Clearly the chambers of $\Delta(P)$ are the maximal chains of P. If P is pure and has length n, then $\Delta(P)$ is pure n-dimensional. The vertex set of $\Delta(P)$ can be partitioned into the rank rows of P. This is clearly a balanced partition and $\Delta(P)$ is a balanced complex.

3. Coxeter Complexes and Buildings

We now review the basic terminology and properties of Coxeter complexes and buildings. Further details, definitions and proofs can be found in [7] and [16]. Throughout this paper (W, S) will denote a Coxeter group and (G, B, N) will denote a group with BN-pair with Weyl group (W, S).

If $w = s_1 s_2 \cdots s_q w \in W$, $s_i \in S$, we call the word $s_1 s_2 \cdots s_q$ in the alphabet S an *expression* for w. The *length* $l(w)$ of $w \in W$ is the least integer q for which an expression $w = s_1 s_2 \cdots s_q$ exists. Such an expression $w = s_1 s_2 \cdots s_q$ of minimum length $q = l(w)$ is said to be *reduced*. Associated with every Coxeter group (W, S) is the *Coxeter diagram*, which is a graph whose vertex set is S and whose edges are the pairs (s_i, s_j) for which s_i and s_j do not commute. Each edge (s_i, s_j) is labeled with the order of $s_i s_j$ in W (when the order is greater than 3). The diagram for $\mathcal{S}_n$, the symmetric group with adjacent transpositions as generators is

Recall that a Coxeter group is irreducible if and only if its diagram is connected. If $D_1, D_2, \ldots, D_k$ are the components of the diagram of (W, S) and (W_i, S_i), $i \in [k]$, is the Coxeter group with diagram D_i, then $W = W_1 \times W_2 \times \cdots \times W_k$ and $S = \biguplus_{i \in [k]} S_i$.

For any subset $J \subset S$, the parabolic subgroup of W generated by J is denoted by W_J. If we fix an ordering of the generators $s_1, s_2, \ldots, s_n$ then we may denote the maximal parabolic subgroup $W_{S - \{s_i\}}$, $i \in [n]$, by $W_{(i)}$.

For any Coxeter group (W, S), the *Coxeter complex* $\Delta(W, S)$ is the simplicial complex whose vertices are cosets $wW_{(i)}$, $i \in [n]$, $w \in W$ and whose chambers are sets $C_w = \{wW_{(i)} \mid i \in [n]\}$, $w \in W$. It is immediate from this definition that $\Delta(W, S)$ is a balanced pure simplicial complex in which the color of a vertex $wW_{(i)}$ is simply i. Note that the C_w are distinct for distinct w and that a coset $wW_{(i)}$ is contained in a chamber C_u if and only if $u \in wW_{(i)}$. In fact the poset of faces of the Coxeter complex is the dual of the poset of all cosets of parabolic subgroups ordered by inclusion. The group W acts on $\Delta(W, S)$ by left translation $w: vW_{(i)} \to wvW_{(i)}$ and this action is color preserving.

We will make use of the following well known properties of groups with BN-pairs.

(3.1) The Weyl group (W, S) is a Coxeter group.

(3.2) (Bruhat decomposition) $G = \biguplus_{w \in W} BwB$.

(3.3) $Bs_1s_2 \cdots s_qB \cdot BwB \subseteq \biguplus_{1 \leq i_1 < \cdots < i_r \leq q} Bs_{i_1}s_{i_2} \cdots s_{i_r}wB$.

(3.4) If $l(u)+l(v)=l(uv)$ then $BuB \cdot BvB = BuvB$.

For any $J \subseteq S$ let G_J be the parabolic subgroup BW_JB. Assume that $s_1, s_2, \ldots, s_n$ is some fixed order of S and let $G_{(i)}$ be $BW_{(i)}B$.

Given any group G with BN-pair, the *building* $\Delta(G, B, N)$ is the simplicial complex whose vertices are cosets $gG_{(i)}$, $i \in [n]$, $g \in G$ and whose chambers are $C_g = \{gG_{(i)} | i \in [n]\}$, $g \in G$. The building $\Delta(G, B, N)$ is a balanced pure simplicial complex in which the color of a vertex $gG_{(i)}$ is i. Note that $C_g = C_{g'}$ if and only if $gB = g'B$ which means that the chambers are indexed by cosets of B. The poset of faces of a building is the dual of the poset of cosets of parabolic subgroups ordered by inclusion. G is a group of color preserving automorphisms of $\Delta(G, B, N)$.

4. Bruhat Order

Let T be the set of conjugates of S, i.e. $T = \{wsw^{-1} | w \in W, s \in S\}$. *Bruhat order* is a partial ordering of W defined as follows: For $u, v \in W$, $u \leq v$ in Bruhat order if there exist $t_1, t_2, \ldots, t_m \in T$ such that $v = ut_1t_2 \cdots t_m$ and $l(ut_1t_2 \cdots t_{i-1}) < l(ut_1t_2 \cdots t_i)$ for $i \in [m]$.

For each $w \in W$, set $T_w = \{t \in T | wt < w\}$. The following properties are well known and their proofs can be found in [7] [9] [18] [19].

(4.1) Strong Exchange Property (Verma). *Let $w \in W$ and $w = s_1s_2 \cdots s_k$. If $wt < w$ then $wt = s_1s_2 \cdots \hat{s}_i \cdots s_k$ (s_i deleted) for some $i \in [k]$. Furthermore if $s_1s_2 \cdots s_k$ is reduced then*

$$T_w = \{s_ks_{k-1} \cdots s_is_{i+1} \cdots s_k | i \in [k]\}.$$

(4.2) Subword Property. *Let $v = s_1s_2 \cdots s_k$ be a reduced expression. Then $u < v$ if and only if there is a reduced expression for u which is a subword of $s_1s_2 \cdots s_k$, i.e. $u = s_{i_1}s_{i_2} \cdots s_{i_j}$ with $1 \leq i_1 < i_2 < \cdots < i_j \leq k$.*

Note that the subword property reveals the left-right symmetry of Bruhat order, which is not apparent from the definition. Hence the strong exchange property and other properties of Bruhat order can be mirrored into a corresponding property.

(4.3) Lifting Property. *If $us > u$ and $vs > v$, then the following are equivalent:*

(a) $u > v$,

(b) $us > v$,

(c) $us > vs$.

Given $w \in W$, we let $\langle w \rangle$ denote some reduced expression for w. It is not difficult to see that the set of generators of $\langle w \rangle$ is independent of the actual expression and depends only on w. We let $\langle u \rangle \langle v \rangle$ be the concatenation of reduced expressions for u and v. Clearly $\langle u \rangle \langle v \rangle$ is reduced if and only if $l(uv) = l(u) + l(v)$.

For $J \subseteq S$, let $W^J = \{w \in W | ws > w$ for all $s \in J\}$ and let ${}^JW = \{w \in W | sw > w$ for all $s \in J\}$. For $I, J \subseteq S$, let ${}^IW^J = {}^IW \cap W^J$. If we have a fixed order of S, $s_1, s_2, \ldots s_n$, then $W^{(i)}$ will denote $W^{S-\{s_i\}}$ and ${}^{(i)}W$ will denote ${}^{S-\{s_i\}}W$.

PROPOSITION 4.4. *Let $I, J \subseteq S$.*
(a) *For any $w \in W$ there is a unique $v \in {}^IW^J$ such that $w = u_1vu_2$ where $u_1 \in W_I$, $u_2 \in W_J$ and $\langle u_1\rangle \langle v\rangle \langle u_2\rangle$ is reduced.*
(b) *${}^IW^J$ is the set of minimum double coset representatives for all double cosets W_IwW_J.*

PROOF. The proof is exercise §1 3) in Bourbaki [7].

COROLLARY 4.5. (A) *Every $w \in W$ can be uniquely factored $w = uv$ where $u \in W^J$ and $v \in W_J$.*
(b) *For any $u \in W^J$ and $v \in W_J$, $\langle u\rangle \langle v\rangle$ is reduced.*
(c) *W^J is the set of minimum left coset representatives modulo W_J.*

Although $W^J = W/W_J$ has no quotient group structure, quotient Bruhat posets, i.e. Bruhat order restricted to W^J (*or* ${}^JW$), are of interest (cf. [5] [9] [14]).

PROPOSITION 4.6. *Let $I, J \subseteq S$ where $I \cap J = \varnothing$ and let $a, d \in W_I$ and $b, c \in W_J$. If $ab = cd$ then all the generators of $\langle a\rangle$ commute with all the generators of $\langle b\rangle$.*

PROOF. The proof is by induction on $l(ab)$. It is easy to see that $\langle a\rangle \langle b\rangle$ and $\langle c\rangle \langle d\rangle$ must be reduced. If $l(ab) = 2$ and $a \neq e$, $b \neq e$, then $a = s_i$, $s_i \in S_I$ and $b = s_j$, $s_j \in S_J$. Since the generators of $\langle c\rangle \langle d\rangle$ are the same as the generators of $\langle a\rangle \langle b\rangle$, $c = s_j$ and $d = s_i$. Hence the result holds for $l(ab) = 2$.

Assume $l(ab) > 2$. This implies that either $l(a) > 1$ or $l(b) > 1$. We may assume without loss of generality that $l(a) > 1$. Let $a = s_1s_2 \cdots s_k$ be reduced and let $t_1 = s_1$ and $t_2 = as_ka^{-1}$. We have that $t_1ab = s_2 \cdots s_kb < ab$. Hence $t_1cd < cd$, which by the strong exchange property implies that $t_1cd = c\hat{d}$ where $\hat{d} < d$. This means that $t_1ab = c\hat{d}$. By induction the generators of $\langle t_1a\rangle$ commute with the generators of $\langle b\rangle$. Similarly since $t_2ab = s_1s_2 \cdots s_{k-1}b < ab$, the generators of $\langle t_2a\rangle$ commute with the generators of $\langle b\rangle$. Since $t_1a = s_2 \cdots s_k$ is reduced and $t_2a = s_1 \cdots s_{k-1}$ is reduced, each s_i, $i \in [k]$ commutes with the generators of $\langle b\rangle$.

If $uv = w$ where $\langle u\rangle \langle v\rangle$ is reduced then we say that u is a *prefix* of w and that v is a *suffix* of w.

PROPOSITION 4.7. *If $w \in {}^JW$, $wa \in {}^JW$ and b is a prefix of a then $wb \in {}^JW$.*

PROOF. Let $a = bc$ and let $v = wa$. Since b is a prefix of a, $\langle b\rangle \langle c\rangle$ is reduced. We have that $v \in {}^JW$ and $vc^{-1} = wb$. If $wb \notin {}^JW$ then there is an $s \in J$ such that $swb < wb$. By the strong exchange property $swb = \hat{w}b$ where $\hat{w} < w$ or $swb = w\hat{b}$ where $\hat{b} < b$. Since $w \in {}^JW$ the former is impossible. Hence $swb = w\hat{b}$. Similarly $svc^{-1} = v\hat{c}^{-1}$ where $\hat{c} < c$. It follows that $w\hat{b} = v\hat{c}^{-1}$ which implies that $w^{-1}v = \hat{b}\hat{c}$. But we also have that $w^{-1}v = a = bc$. Hence $\hat{b}\hat{c} = bc$. This is impossible since $\langle b\rangle\langle c\rangle$ is reduced.

5. QUOTIENT COMPLEXES

For $J \subseteq S$, the subcomplex of $\Delta(W, S)$ whose vertex set is $\{wW_{(i)} | w \in {}^JW, i \in [n]\}$ and whose chambers are the sets $C_w = \{wW_{(i)} | i \in [n]\}$, $w \in {}^JW$ will be denoted by $\Delta({}^JW, S)$.

These subcomplexes arise in the work of Garsia and Stanton [11] in which they consider quotient complexes. If H is a group acting on a simplicial complex Δ then the *quotient complex* corresponding to H, is denoted Δ/H and is defined to be a poset whose elements are the orbits of faces of Δ under the action of H. The order relation is: $\sigma_1 < \sigma_2$ if a face $f \in \sigma_1$ is contained in a face $g \in \sigma_2$. It is important to note that Δ/H does not have to be simplicial (i.e. the poset of faces of a simplicial complex). Garsia and Stanton consider, in particular, the action of W_J on $\Delta(W, S)$ by left translation. It turns out that in this case the quotient complex is simplicial. The following correspondence was observed by Garsia.

THEOREM 5.1. *The poset of faces of* $\Delta({}^JW, S)$ *is isomorphic to* $\Delta(W, S)/W_J$.

PROOF. Recall that the poset of faces of $\Delta(W, S)$ is isomorphic to the dual of the poset of cosets of parabolic subgroups. Hence we may refer to a coset wW_I, as a face of $\Delta(W, S)$.

For any face wW_I of $\Delta(W, S)$, let $\sigma(wW_I)$ denote its orbit. Since W_J acts by left translation, faces of $\sigma(wW_I)$ are of the form uwW_I, $u \in W_J$. Let $\phi : \Delta(W, S)/W_J \to \Delta({}^JW, S)$ be defined by $\phi(\sigma(wW_I)) = vW_I$ where v is the minimum element (under Bruhat order) of the double coset W_JwW_I. Clearly ϕ is a bijection. We must now show that ϕ is order preserving. Let $\sigma(w'W_K) > \sigma(wW_I)$. Then $K \subseteq I$ and we may assume that $w' \in wW_I$. Let v be the minimum element of W_JwW_I and v' be the minimum element of $W_Jw'W_K$. Since $\phi(\sigma(wW_I)) = vW_I$ and $\phi(\sigma(w'W_K)) = v'W_K$, we must show that $vW_I \supseteq v'W_K$ which is equivalent to showing that $v' \in vW_I$. We have that

$$\begin{aligned} v' \in W_Jw'W_K &\subseteq W_JwW_IW_K \\ &= W_JwW_I \\ &= W_JvW_I. \end{aligned}$$

By Proposition 4.4 $v' = u_1vu_2$ where $u_1 \in W_J$, $u_2 \in W_I$ and $\langle u_1\rangle\langle v\rangle\langle w_2\rangle$ is reduced. Since $v' \in {}^JW$ it follows that $u_1 = e$. Thus, $v' \in vW_I$.

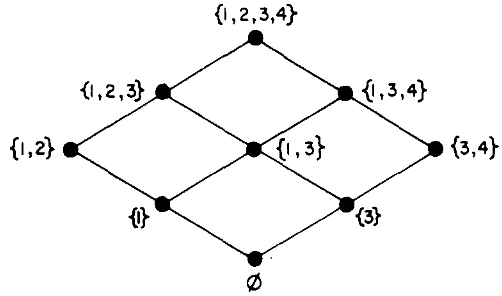
Conversely if the face $v'W_K \in \Delta({}^JW, S)$ contains the face $vW_I \in \Delta({}^JW, S)$ then $\sigma(v'W_K) > \sigma(vW_I)$. Hence ϕ and its inverse are order preserving.

We shall refer to the subcomplex $\Delta({}^JW, S)$ as a *quotient Coxeter complex.*

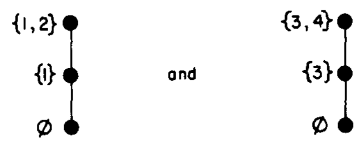
Now let (G, B, N) be a group with BN-pair and let (W, S) be its Weyl group. Since G_J acts on $\Delta(G, B, N)$ by left translation, we get a quotient complex $\Delta(G, B, N)/G_J$. Since this can easily by shown to be isomorphic to $\Delta(W, S)/W_J$, $\Delta(G, B, N)/G_J$ gives nothing new. To obtain a simplicial complex analogous to $\Delta({}^JW, S)$ we let ${}^JG = B\,{}^JWB$ and let $\Delta({}^JG, B, N)$ be the subcomplex of $\Delta(G, B, N)$ whose chambers are sets $C_g = \{gG_{(i)} \mid i \in [n]\}$, $g \in {}^JG$. We shall refer to $\Delta({}^JG, B, N)$ as a *quotient building*, although it is not a quotient in the usual sense.

5.1. EXAMPLE. Let W be the symmetric group $\mathcal{S}_n$ and let $S = \{s_1, s_2, \ldots, s_{n-1}\}$ be the set of adjacent transpositions where $s_i = (i, i+1)$. The Coxeter complex $\Delta(W, S)$ is the order complex of the proper part of the Boolean algebra, B_n i.e. the lattice of subsets of $[n]$ ordered by inclusion. Since a maximal chain of B_n is of the form $\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = [n]$ where $|A_i - A_{i-1}| = 1$, the map $\gamma(A_1 \subset \cdots \subset A_{n-1}) = (A_1 - A_0, A_2 - A_1, \ldots, A_n - A_{n-1})$ is a bijection between the chambers of $\Delta(W, S)$ and permutations in $\mathcal{S}_n$. It can be easily checked that $\gamma(C_w) = w$.

Now let $n = 4$ and $J = \{(12)(34)\}$. Then ${}^JW = \{(1234), (1324), (1342), (3124), (3142), (3412)\}$. The subposet of B_4 whose maximal chains correspond to these permutations is:



Note that this poset is the product of the posets



Hence $\Delta({}^JW, S)$ is the order complex of the proper part of a product of chains.

THEOREM 5.2. *If $W=\mathcal{S}_n$, and S is the set of adjacent transpositions, and $J\subseteq S$, then $\Delta({}^JW, S)$ is the order complex of the proper part of a product of chains.*

PROOF. For $i\in[n-1]$, let s_i be the transposition $(i, i+1)$ and let $S-J=\{s_{i_1}, s_{i_2}, \ldots, s_{i_j}\}$ where $1\leq i_1<i_2<\cdots<i_j\leq n-1$. Let $N_0=\{1,2,\ldots,i_1\}$, $N_1=\{i_1+1, i_1+2,\ldots,i_2\},\ldots,$ $N_j=\{i_j+1, i_j+2,\ldots,n\}$. It is easy to see that ${}^JW$ consists of shuffles of $N_0, N_1,\ldots,N_j$, i.e. permutations for which the elements of each N_i appear in their natural order.

For $T\subseteq[n]$, let B_T denote the lattice of subsets of T ordered by inclusion. Clearly $B_n\simeq B_{N_0}\times B_{N_1}\times\cdots\times B_{N_j}$. In each B_{N_k}, let c_k be the maximal chain $\phi\subset\{i_k+1\}\subset\{i_k+1, i_k+2\}\subset\cdots\subset\{i_k+1, i_k+2,\ldots,i_{k+1}\}$ (where $i_0=0$ and $i_{j+1}=n$). The direct product $c_0\times c_1\times\cdots\times c_j$ is isomorphic to a subposet X of B_n. The map γ applied to any maximal chain of $\bar{X}$ clearly yields a shuffle of $N_0, N_1,\ldots,N_j$ and conversely any shuffle is the image of a maximal chain of $\bar{X}$. Hence the maximal chains of $\bar{X}$ are the chambers of $\Delta({}^JW, S)$ and consequently $\Delta({}^JW, S)$ is the order complex of $\bar{X}$.

It follows from Theorem 5.2 that for $W=\mathcal{S}_n$, $\Delta({}^JW, S)$ is the order complex of the proper part of a lexicographically shellable lattice since chains are lexicographically shellable and products of lexicographically shellable posets are lexicographically shellable (cf [2]). In the subsequent sections we prove this for general Coxeter groups with linear diagram.

The subcomplex $\Delta(W^J, S)$ which can be defined analogously to $\Delta({}^JW, S)$ does not have the properties just mentioned above. For example if $W=\mathcal{S}_4$ and $J=\{(2,3)\}$ then $\Delta(W^J, S)$ is not the chain complex of a poset. However, these subcomplexes are significant (cf. [3]).

6. ORDER COMPLEXES

In this section we characterize those quotient Coxeter complexes and buildings that are order complexes. We also show that they are in fact order complexes of lattices. The following lemmas will be used here and in subsequent sections.

LEMMA 6.1. *Let $I, J, K \subseteq S$. If*
(a) $wW_I \cap {}^JW \neq \emptyset$,
(b) $uW_K \cap {}^JW \neq \emptyset$,
(c) $wW_I \cap uW_K \neq \emptyset$,
then $wW_I \cap uW_K \cap {}^JW \neq \emptyset$.

PROOF. We may assume that $w, u \in {}^JW$. It follows from (c) that $w\alpha = u\beta$ for some $\alpha \in W_I$, $\beta \in W_K$. By Corollary 4.5 we may factor $\alpha = \alpha_1\alpha_2$ where $\alpha_1 \in W^K$, $\alpha_2 \in W_K$. Now we have $w\alpha_1 = u\beta\alpha_2^{-1}$ which implies that $w\alpha_1 \in wW_I \cap uW_K$. By Corollary 4.5, α_1 is a prefix of $\alpha_1\alpha_2\beta^{-1} = \alpha\beta^{-1}$. Since $w \in {}^JW$ and $w\alpha\beta^{-1} \in {}^JW$, applying Proposition 4.7 gives $w\alpha_1 \in {}^JW$. Hence $w\alpha_1 \in wW_I \cap uW_K \cup {}^JW$.

LEMMA 6.2. *Let $I, J, K \subseteq S$. If*
(a) $gG_I \cap {}^JG \neq \emptyset$,
(b) $fG_K \cap {}^JG \neq \emptyset$,
(c) $gG_I \cap fG_K \neq \emptyset$,
then $gG_I \cap fG_K \cap {}^JG \neq \emptyset$.

PROOF. Let $g \in G^I$ and $f \in G^K$. We first show that $g, f \in {}^JG$. By (a) $gb\alpha b' \in {}^JG$ for some $\alpha \in W_I$, $b, b' \in B$. By (3.4) if $g \in BwB$ then $gb\alpha b' \in Bw\alpha B$. By Bruhat decomposition (3.2), $w\alpha \in {}^JW$. It follows from Proposition 4.7 that $w \in {}^JW$. Hence $g \in {}^JG$. Similarly $f \in {}^JG$.

It follows from (c) that $gb_1\alpha b_1' = fb_2\beta b_2'$ where $b_1, b_1', b_2, b_2' \in B$ and $\alpha \in W_I$, $\beta \in W_K$. Let $g \in BwB$ and $f \in BuB$. Then $w \in {}^JW^I$ and $u \in {}^JW^K$. It follows from (3.4) and (3.2) that $w\alpha = u\beta$. Now let $\alpha = \alpha_1\alpha_2$ where $\alpha_1 \in W^K$ and $\alpha_2 \in W_K$. Just as in the proof of Lemma 6.1 $w\alpha_1 \in {}^JW$. Hence by (3.4) $gb_1\alpha_1 \in {}^JG$. Since $gb_1\alpha_1 = fb_2\beta b_2'(\alpha_2 b_1')^{-1}$, we also have that $gb_1\alpha_1 \in gG_I \cap fG_K$.

THEOREM 6.3. *For any $J \subseteq S$, $\Delta({}^JW, S)$ is an order complex if and only if the components of the Coxeter diagram that intersect $S - J$ are linear.*

PROOF. ($\Rightarrow$) Let P be a poset whose order complex is $\Delta({}^JW, S)$. P is clearly a pure poset of length $n-1$, where $n = |S|$. The rank rows of P form a balanced partition of $\Delta({}^JW, S)$. To show that a rank row of P is the set of cosets of $W_{S-\{s\}}$ for some $s \in S$, it suffices to show that there is only one balanced partition of $\Delta({}^JW, S)$. Assume there is a coloring of the vertices of C_e. We claim that this coloring forces a coloring of C_w for all $w \in {}^JW$. Let $w = s_1s_2 \cdots s_k$ be reduced and $w_i = s_1s_2 \cdots s_i$ for $i = 0, \ldots, k$. By Proposition 4.7 $w_i \in {}^JW$. Since $|C_{w_i} \cap C_{w_{i-1}}| = n-1$, any coloring of $C_{w_{i-1}}$ determines a coloring of C_{w_i}. Hence there is only one way to color C_w after coloring $C_{w_0} = C_e$. Thus the rank rows of P are the sets $\{wW_{S-\{s\}} | w \in {}^JW\}$, $s \in S$. We may now order the elements of S, $s_1s_2, \ldots, s_n$ corresponding to the rank of $wW_{S-\{s\}}$ in P. The order relation in P is: $wW_{(i)} < uW_{(j)}$ if and only if $wW_{(i)} \cap uW_{(j)} \cap {}^JW \neq \emptyset$ and $i < j$.

Let D be the Coxeter diagram for (W, S). Let s_i and s_j belong to a component of D that intersects $S - J$. We must show that if $|i-j| > 1$ then s_i and s_j commute. Assume that s_i and s_j do not commute. Let $r_0, r_1, \ldots, r_m, s_i, s_j$ be a path in D such that $r_0 \in S - J$ and the vertices of the path are distinct.

Let $u = r_0r_1 \cdots r_m$. We will show that $u \in {}^JW$. If not then $su < u$ for some $s \in J$. Clearly $s = r_i$ for some $i \in [m]$. By the strong exchange property $u = r_ir_0 \cdots \hat{r}_i \cdots r_m$. Hence $r_ir_0 \cdots r_{i-1} = r_0 \cdots r_{i-1}r_i$. It follows from Proposition 4.6 that r_i commutes with r_{i-1}. This contradicts the fact that $r_0, r_1, \ldots, r_m$ is a path. Hence $u \in {}^JW$. The same argument shows that $us_is_j \in {}^JW$.

We may assume without loss of generality that $i+1<j$. Let k be such that $i<k<j$. Consider the cosets $W_{(i)}$, $uW_{(k)}$, $us_is_jW_{(j)}$. Since $u \in W_{(i)} \cap uW_{(k)} \cap {}^JW$, it follows that $W_{(i)} < uW_{(k)}$. Since $us_is_j \in uW_{(k)} \cap us_is_jW_{(j)} \cap {}^JW$ we also have that $uW_{(k)} < us_is_jW_{(j)}$. By transitivity $W_{(i)} < us_is_jW_{(j)}$. Hence $W_{(i)} \cap us_is_jW_{(j)} \neq \varnothing$. This implies that

$$us_is_jw \in W_{(i)} \quad \text{for some } w \in W_{(j)}. \tag{6.1}$$

The argument that was used to show that $u \in {}^JW$ can be used to show that $us_is_j \in W^{(j)}$. This implies that $\langle u\rangle s_is_j\langle w\rangle$ is a reduced expression contradicting (6.1).

($\Leftarrow$)*Case* 1. Assume the Coxeter diagram D is connected. Let $s_1, s_2, \ldots, s_n$ be one of the two linear orderings of S induced by D. We define an ordering on the vertices of $\Delta({}^JW, S)$ as follows: $uW_{(i)} < vW_{(j)}$ if $i<j$ and $uW_{(i)} \cap vW_{(j)} \cap {}^JW \neq \varnothing$. To show that this defines a partial order, we need only establish transitivity. To show that the order complex of the partial order is $\Delta({}^JW, S)$, we must be sure that the order complex does not introduce additional faces that are not in $\Delta({}^JW, S)$. The following lemma insures this and also establishes the transitivity of $<$.

LEMMA 6.4. *If the sequence* $w_\perp W_{(i_1)}, w_2W_{(i_2)}, \ldots, w_mW_{(i_m)}$ *satisfies:*
(a) $i_1 < i_2 < \cdots < i_m$
(b) ${}^JW \cap w_jW_{(i_j)} \cap w_{j+1}W_{(i_{j+1})} \neq \varnothing, j \in [m-1]$
then ${}^JW \cap \bigcap_{j=1}^m w_jW_{(i_j)} \neq \varnothing$.

PROOF. The proof is by induction on m and is trivial when $m=2$. Assume $m>2$. By induction there exists u, v such that

$$u \in {}^JW \cap \bigcap_{j=1}^{m-1} w_jW_{(i_j)}$$

and

$$v \in {}^JW \cap \bigcap_{j=m-1}^{m} w_jW_{(i_j)}.$$

It follows that $u \in vW_{(i_{m-1})}$. Since D is linear, $u\beta = v\alpha$ where $\alpha \in W_{\{s_1,\ldots,s_{i_{m-1}-1}\}}$ and $\beta \in W_{\{s_{i_{m-1}+1},\ldots,s_n\}}$. Clearly $v\alpha \in \bigcap_{j=m-1}^m vW_{(i_j)} = vW_I$ and $u\beta \in \bigcap_{j=1}^{m-1} uW_{(i_j)} = uW_K$ where $I = S - \{s_{i_{m-1}}, s_{i_m}\}$ and $K = S - \{s_{i_1}, s_{i_2}, \ldots, s_{i_{m-1}}\}$. It follows from Lemma 6.1 that

$${}^JW \cap \bigcap_{j=1}^{m} w_jW_{(i_j)} = {}^JW \cap uW_K \cap vW_I \neq \varnothing.$$

Case 2. Assume that D is disconnected. Let $D_1, D_2, \ldots, D_k$ be the components of D and let (W_i, S_i), $i \in [k]$ be the Coxeter group with diagram D_i. It is well known and easy to show that $\Delta(W, S) = \Delta(W_1, S_1) \oplus \Delta(W_2, S_2) \oplus \cdots \oplus \Delta(W_k, S_k)$, where $\oplus$ denotes the join which is defined as follows: If Δ_1, Δ_2 are simplicial complexes then $\Delta_1 \oplus \Delta_2$ is the simplicial complex whose faces are the unions of faces of Δ_1 and Δ_2. It is also easy to show that $\Delta({}^JW, S) = \Delta({}^JW_1, S_1) \oplus \Delta({}^JW_2, S_2) \oplus \cdots \oplus \Delta({}^JW_k, S_k)$. For posets P and Q let $P \oplus Q$ denote the ordinal sum of P and Q with order relation $x<y$ if $x, y \in P$, $x<y$ in P or $x, y \in Q$, $x<y$ in Q or $x \in P$, $y \in Q$. Clearly the order complex of $P \oplus Q$ is the join of the order complexes of P and Q. Hence if $\Delta({}^JW_i, S_i)$ $i \in [k]$ is the order complex of a poset so is $\Delta({}^JW, S)$. If $(S-J) \cap S_i \neq \varnothing$ then D_i is linear and by Case 1, $\Delta({}^JW_i, S_i)$ is the order complex of a poset. If $S_i \subseteq J$ then ${}^JW_i = e$ and $\Delta({}^JW_i, S_i)$ is the order complex of a chain. Hence $\Delta({}^JW, S)$ is the order complex of a poset. This completes the proof of Theorem 6.3.

THEOREM 6.5. *For any* $J \subseteq S$, $\Delta({}^JG, B, N)$ *is an order complex if and only if the components of the Coxeter diagram that intersect* $S-J$ *are linear.*

PROOF. ($\Rightarrow$) The proof of Theorem 6.3 works with slight modification. The cosets being compared are $G_{(i)}$, $uG_{(k)}$, $us_is_jG_{(j)}$. Transitivity yields $G_{(i)} \cap us_is_jG_{(j)} \neq \varnothing$ which implies $us_is_jg \in G_{(i)}$, $g \in G_{(j)}$. Since $g \in G_{(j)}$, $g \in BwB$ where $w \in W_{(j)}$. Just as in Theorem 6.3 $\langle u\rangle s_is_j\langle w\rangle$ is reduced. Hence by (3.4) $us_is_jg = bus_is_jwb'$. This implies that $us_is_jw \in W_{(i)}$ which is a contradiction.

($\Leftarrow$) We modify the proof of Theorem 6.3 in the obvious way and use Lemma 6.2.

Theorem 6.5 can also be proved by using Theorem 6.3. The special case of Theorems 6.3 and 6.5 in which $J = \varnothing$ was obtained jointly with Björner. Theorem 6.3 for $J = \varnothing$ is also included in the work of Vince [20]. Tits [16] and Surowski [15] also established the sufficiency of linear diagrams for Coxeter complexes and buildings to be order complexes. Also in [16], Tits using a different approach from ours, establishes Theorem 6.6 below for $J = \varnothing$ and Theorem 6.8.

Let us now assume that (W, S) is irreducible with linear diagram and that S is ordered $s_1, s_2, \ldots, s_n$ so that s_i and s_j commute if and only if $|i-j| \neq 1$. Let P be a poset whose order complex is $\Delta({}^JW, S)$. To obtain $\hat{P}$ we include two additional cosets W_0 and W_{n+1} as elements of P, where W_0 and W_{n+1} are distinct copies of W. The order relation on $\hat{P}$ is simply: For $i, j = 0, 1, \ldots, n+1$, $wW_i < uW_j$ if $wW_i \cap uW_j \cap {}^JW \neq \varnothing$ and $i<j$. Clearly $W_0 = \hat{0}$ and $W_{n+1} = \hat{1}$ in $\hat{P}$.

THEOREM 6.6. *If (W, S) is irreducible and $\Delta({}^JW, S)$ is the order complex of poset P, then $\hat{P}$ is a lattice.*

In proving Theorem 6.6 we use the following lemma.

LEMMA 6.7. *If $wW_I \cap {}^JW \neq \varnothing$ and $uW_H \cap {}^JW \neq \varnothing$ then there exists $w' \in wW_I \cap {}^JW$ and $u' \in uW_H \cap {}^JW$ such that $(w')^{-1}u' \in IW^H$.*

PROOF. Assume that $u, w \in {}^JW$. Let $\gamma = w^{-1}u$. By Proposition 4.4 we can factor $\gamma = \gamma_1\gamma_2\gamma_3$ so that $\gamma_1 \in W_I$, $\gamma_2 \in {}^IW^H$, $\gamma_3 \in W_H$ and $\langle\gamma_1\rangle\langle\gamma_2\rangle\langle\gamma_3\rangle$ is reduced. Thus $u = w\gamma_1\gamma_2\gamma_3$. Let $u' = u\gamma_3^{-1}$ and $w' = w\gamma_1$. Clearly $(w')^{-1}u' = \gamma_2 \in {}^IW^H$, $u' \in uW_H$ and $w' \in wW_I$. Since γ_1 is a prefix of γ and $w, w\gamma \in {}^JW$, it follows from Proposition 4.7 that $w' \times w\gamma_1 \in {}^JW$. Similarly $u' \in {}^JW$.

PROOF OF THEOREM 6.6. To prove that $\hat{P}$ is a lattice it suffices to show that any $wW_{(i)}$ and $uW_{(h)}$ have a meet, since by duality they then must also have a join. Assume that $h \leqslant i$ and let γ be the unique minimal element of $W_{(i)}w^{-1}uW_{(h)}$. By Lemma 6.7 we can assume that $u, w \in {}^JW$ and $w^{-1}u = \gamma$. Let $k = \max\{m \mid \gamma \in W_{(m)}, 0 \leqslant m \leqslant h\}$. We claim that $wW_{(k)}$ $(= uW_{(k)})$ is the meet of $wW_{(i)}$ and $uW_{(h)}$.

We prove this claim by induction on $\rho(wW_{(i)}) + \rho(uW_{(h)}) = i+h$. If $i+h=0$ the claim is trivial. Assume $i+h>0$. Take any coset less than $wW_{(i)}$ and $uW_{(h)}$. Such a coset must contain elements w' and u' where $w'W_{(i)} = wW_{(i)}$, $u'W_{(h)} = uW_{(h)}$ and $u', w' \in {}^JW$. The coset is therefore of the form $w'W_{(j)}(= u'W_{(j)})j \leqslant h$. We must now show that

$$wW_{(k)} \geqslant w'W_{(j)}. \tag{6.2}$$

We have the following

$$\begin{aligned} w' &= w\alpha, && \text{where } \alpha \in W_{(i)}, \\ u' &= u\delta, && \text{where } \delta \in W_{(h)}, \\ u' &= w'\beta && \text{where } \beta \in W_{(j)}, \\ u &= w\gamma && \text{where } \gamma \in W_{(k)} \cap {}^{(i)}W^{(h)}. \end{aligned} \tag{6.3}$$

Case 1. Assume that $l(\gamma)=1$. Since $\gamma\in {}^{(i)}W^{(h)}$ it follows that $\gamma=s_i$, $i=h$ and $k=i-1$. From (6.3) we have $\alpha\beta=s_i\delta$. Thus $\beta\in W_{(i)}s_iW_{(i)}$. By Proposition 4.4, $\beta=\beta_1s_i\beta_2$ where $\beta_1,\beta_2\in W_{(i)}$ and $\langle\beta_1\rangle s_i\langle\beta_2\rangle$ is reduced. Clearly $\alpha\beta_1s_i=s_i\delta\beta_2^{-1}$, $\alpha\beta_1\in W_{(i)}$, and $\delta\beta_2\in W_{(i)}$. By Proposition 4.6, s_i commutes with the generators in $\langle\alpha\beta_1\rangle$. Consequently $\alpha\beta_1\in W_{(i-1)}$. This implies that

$$w\alpha\beta_1\in wW_{(i-1)}=wW_{(k)}.$$

Since $\beta_1<\beta$ it follows that $\beta_1\in W_{(j)}$, which implies that

$$w\alpha\beta_1=w'\beta_1\in w'W_{(j)}.$$

Hence $w'W_{(j)}\cap wW_{(k)}\neq\varnothing$. By Lemma 6.1 and since $j\leqslant k$, (6.2) holds.

Case 2. Assume that $l(\gamma)>1$. Since $\gamma\in {}^{(i)}W^{(h)}$, γ is of the form $s_i\gamma's_h$ where $s_i\langle\gamma'\rangle s_h$ is reduced. It follows from (6.3) that

$$\alpha\beta=\gamma\delta.$$

Let $t\in T_\gamma$ be such that $t\gamma=s_i\gamma'$. Since $\gamma\in W^{(h)}$ and $\delta\in W_{(h)}$, $\langle\gamma\rangle\langle\delta\rangle$ is reduced. Hence $t\gamma\delta<\gamma\delta$ which means that $t\alpha\beta<\alpha\beta$. By the strong exchange property either,

$$t\alpha\beta=\hat{\alpha}\beta,\quad \text{where } \hat{\alpha}<\alpha, \tag{6.4}$$

or

$$t\alpha\beta=\alpha\hat{\beta},\quad \text{where } \hat{\beta}<\beta. \tag{6.5}$$

If (6.4) holds then $t\in W_{(i)}$ which implies that $t\gamma>\gamma$. Hence (6.5) holds.

Let $v=wt\gamma$. Since $w, w\gamma\in {}^JW$ and $t\gamma$ is a prefix of γ, by Proposition 4.7 we have $v=wt\gamma\in {}^JW$. Hence the coset $vW_{(h)}$ is an element of $\hat{P}$. Since $v=wt\gamma\in wW_{(k)}$, we have that

$$vW_{(h)}\geqslant wW_{(k)}(=uW_{(k)}). \tag{6.6}$$

Clearly $v\delta\in vW_{(h)}$ and $v\delta=wt\gamma\delta=wt\alpha\beta=w\alpha\hat{\beta}=w'\hat{\beta}\in w'W_{(j)}$. Hence $vW_{(h)}\cap w'W_{(j)}\neq\varnothing$. By Lemma 6.1

$$vW_{(h)}\geqslant w'W_{(j)}(=u'W_{(j)}). \tag{6.7}$$

We may now apply case 1 to the cosets $vW_{(h)}$ and $uW_{(h)}$ since $u^{-1}v=s_h$. Let x be the meet of $vW_{(h)}$ and $uW_{(h)}$. It follows from (6.6) and (6.7) that

$$x\geqslant uW_{(k)}\quad\text{and}\quad x\geqslant u'W_{(j)}. \tag{6.8}$$

Since $\rho(x)<h$, by induction x and $wW_{(i)}$ have a meet y. It follows from (6.8) that

$$y\geqslant wW_{(k)}\quad\text{and}\quad y\geqslant w'W_{(j)}. \tag{6.9}$$

Since $y\leqslant x\leqslant uW_{(h)}$ and $y\leqslant wW_{(i)}$ it follows that $u''\in y\cap uW_{(h)}$ and $w''\in y\cap wW_{(i)}$ for some u'', w''. Hence $(w'')^{-1}u''\in W_{(i)}w^{-1}uW_{(h)}=W_{(i)}\gamma W_{(h)}$. By Proposition 4.4, $(w'')^{-1}u''>\gamma$. Hence $(w'')^{-1}u''\notin W_m$ for all $m=k+1,k+2,\ldots,h$. It follows that $\rho(y)\leqslant k$. This together with (6.9) implies that $y=wW_{(k)}$ and that $wW_{(k)}\geqslant w'W_{(j)}$.

We have not been able to obtain Theorem 6.6 for quotient buildings except when $J=\varnothing$. In this case we have the following:

THEOREM 6.8. *If (W,S) is irreducible and $\Delta(G,B,N)$ is the order complex of poset P then $\hat{P}$ is a lattice.*

PROOF. We let g and f play the role of w and u, respectively, in the proof of Theorem 6.6. We show that cosets $gG_{(i)}$ and $fG_{(h)}$, where $h\leqslant i$, have a meet. We claim that g and

f can be chosen so that $g^{-1}f \in {}^{(i)}W^{(h)}$. Indeed if $f = gb_1\gamma b_2$ where $\gamma \in W$ and $b_1, b_2 \in B$, then $fb_2^{-1} = gb_1\gamma$. Since $fb_2^{-1}G_{(h)} = fG_{(h)}$ and $gb_1G_{(i)} = gG_{(i)}$, we can assume that $f = g\gamma$. Since γ can be factored into $\gamma_1\gamma_2\gamma_3$ where $\gamma_1 \in W_{(i)}$, $\gamma_2 \in {}^{(i)}W^{(h)}$, $\gamma_3 \in W_{(h)}$, it follows that $f\gamma_3^{-1} = (g\gamma_1)\gamma_2$. Now we have $f\gamma_3^{-1}G_{(h)} = fG_{(h)}$ and $g\gamma_1G_{(i)} = gG_{(i)}$. Hence we can assume that $\gamma \in {}^{(i)}W^{(h)}$.

We shall use induction on $(i+h, l(\gamma))$ ordered lexicographically to show that $gG_{(i)}$ and $fG_{(h)}$ have meet, $gG_{(k)}(=fG_{(k)})$, where k is defined as in the proof of Theorem 6.6. Let $g'G_{(j)}$ $(=f'G_{(j)})$ be a coset less than both $gG_{(i)}$ and $fG_{(h)}$. It follows that

$$\begin{aligned} g' &= gb_1\alpha b_1', && \text{where } \alpha \in W_{(i)},\\ f' &= fb_2\delta b_3, && \text{where } \delta \in W_{(h)},\\ f' &= g'b_4'\beta b_4, && \text{where } \beta \in W_{(j)},\\ f &= g\gamma && \text{where } \gamma \in W_{(k)} \cap {}^{(i)}W^{(h)}, \end{aligned} \tag{6.10}$$

and where $b_1, b_1', b_2, b_3, b_4', b_4 \in B$. We claim that g' can be chosen so that $b_1' = b_4' = e$. Indeed by (6.10) and (3.3) we have, $f' = gb_1\alpha b_1'b_4'\beta b_4 = gb\alpha\hat{\beta}b'$ where $\hat{\beta} < \beta$ and $b,b' \in B$. Choose $g' = gb\alpha$ and call b,b_1, call b',b_4 and call $\hat{\beta},\beta$. We now have

$$\begin{aligned} g' &= gb_1\alpha && \text{where } \alpha \in W_{(i)}\\ f' &= fb_2\delta b_3 && \text{where } \delta \in W_{(h)}\\ f' &= g'\beta b_4 && \text{where } \beta \in W_{(j)}\\ f &= g\gamma && \text{where } \gamma \in W_{(k)} \cap {}^{(i)}W^{(h)}. \end{aligned}$$

It follows from this, (3.4) and Bruhat decomposition that

$$\alpha\beta = \gamma\delta.$$

Case 1. Assume that $l(\gamma) = 1$. Let β_1, β_2 be as in the proof of Theorem 6.6. We can show that

$$gb_1\alpha\beta_1 \in gG_{(k)} \cap g'G_{(j)},$$

which completes the proof for case 1.

Case 2. Assume that $l(\gamma) > 1$. Let t be as in the proof of Theorem 6.6 and now let

$$v = gb_1t\gamma.$$

Just as in the proof of Theorem 6.6 we can show that

$$vG_{(h)} \geq gG_{(k)}(=fG_{(k)})$$

and (6.11)

$$vG_{(h)} \geq g'G_{(j)}(=f'G_{(j)}).$$

We have $(gb_1)^{-1}v = t\gamma$. Let $t\gamma = \gamma_1\gamma_2$ where $\gamma_1 \in W^{(h)}$, $\gamma_2 \in W_{(h)}$. Since γ_1 is a prefix of $t\gamma$ which is in turn a prefix of γ, it follows that $\gamma_1 \in {}^{(i)}W^{(h)}$. Hence we have

$$(gb_1)^{-1}(v\gamma_2^{-1}) = \gamma_1 \in {}^{(i)}W^{(h)}.$$

Since $l(\gamma_1) < l(\gamma)$ we can apply the induction hypothesis to the cosets $gb_1G_{(i)}$ and $v\gamma_2^{-1}G_{(h)}$. Let x be the meet of $gb_1G_{(i)}$ and $v\gamma_2^{-1}G_{(h)}$. Since $gb_1G_{(i)} = gG_{(i)}$ and $v\gamma_2^{-1}G_{(h)} = vG_{(h)}$, x is the meet of $gG_{(i)}$ and $vG_{(h)}$. (Note that if $\gamma_1 = e$ then $x = vG_{(h)}$ and $h < i$.) By (6.11) we have

$$x \geqslant gG_{(k)}(=fG_{(k)})$$

and (6.12)

$$x \geqslant g'G_{(j)}(=f'G_{(j)}).$$

Since $\rho(x) < i$, by induction x and $fG_{(h)}$ have a meet y. It follows from (6.12) that

$$y \geqslant gG_{(k)}(=fG_{(k)})$$

and (6.13)

$$y \geqslant g'G_{(j)}(=f'G_{(j)}).$$

Since $y \leqslant x \leqslant gG_{(i)}$ and $y \leqslant fG_{(h)}$, it follows that $g'' \in y \cap gG_{(i)}$ and $f'' \in y \cap fG_{(h)}$ for some g'' and f''. If $\rho(y) = r$ then $y = f''G_{(r)} = g''G_{(r)}$. Consequently $(g'')^{-1}f'' \in G_{(r)}$ which means that $(g'')^{-1}f'' \in B\gamma'B$ where $\gamma' \in W_{(r)}$. We also have that $(g'')^{-1}f'' \in G_{(i)}g^{-1}fG_{(h)} = G_{(i)}\gamma G_{(h)}$. By (3.2), (3.3) and Proposition 4.4, $\gamma' \geqslant \gamma$. Hence $\gamma \in W_{(r)}$. It follows that $\rho(y) = r \leqslant k$, which together with (6.13) implies that $gG_{(k)} \geqslant g'G_{(j)}$.

7. Lexicographical Shellability

Shellability is a property of simplicial complexes that can be extended to posets by considering the order complex of the poset. The notion of lexicographical shellability which was introduced by Björner [2], is a special type of shellability which is defined only for graded posets. For an expository treatment of shellable posets see [4]. In [3] Björner shows that Coxeter complexes and buildings are shellable. He shows that the shelling orders on the chambers correspond to linear extensions of Bruhat order on W. In [11] Garsia and Santon exend Björner's result to quotients. Since any poset P, whose order complex is a quotient Coxeter complex or quotient building is shellable, it seems natural to ask whether $\hat{P}$ is in fact lexicographically shellable. We answer this question in Theorems 7.3 and 7.6.

A graded poset is lexicographically shellable if its Hasse diagram admits a certain type of edge labeling which is described in [6]. Many important classes of posets are lexicographically shellable (cf. [2][4][5][6][12]). There are two different versions of lexicographical shellability; EL-(edge lexicographical) shellability and CL-(chain lexicographical) shellability (the more general version). The version that we deal with here is CL-shellability. In [6] a recursive formulation of CL-shellability is presented. We use this formulation of CL-shellability throughout this section.

Definition 7.1. A graded poset P is said to *admit a recursive atom ordering* if the length of P is 1 or if the length of P is greater than 1 and there is a well ordering Ω of the atoms of P which satisfies:

(a) For any atom a of P, $[a, \hat{1}]$ admits a recursive atom ordering in which the atoms of $[a, \hat{1}]$ that come first in the ordering are those that cover some a' where a' precedes a in Ω.

(b) For any atoms a,a' of P such that a' precedes a in Ω, if $a,a' < y$ then there is an atom a'' and an element z such that a'' precedes a in Ω and $a'', a \rightarrow z \leqslant y$. (Recall that $\rightarrow$ means 'covered by'.)

If Ω is a well ordering of the atoms of P that satisfies (a) and (b) then Ω is said to be a *recursive atom ordering*.

An example of a poset which admits a recursive atom ordering is given in Figure 1(a) and one which does not admit a recursive atom ordering is given in Figure 1(b).

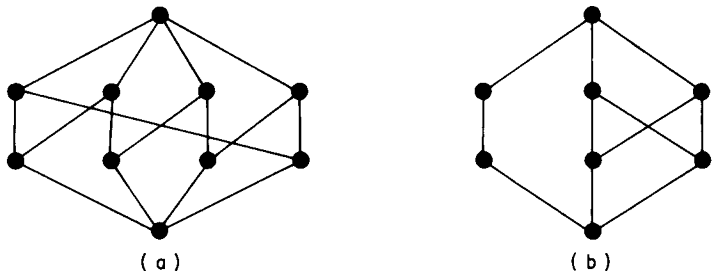


FIGURE 1.

Although shellability is traditionally defined only for *finite* pure simplicial complexes and lexicographical shellability only for *finite* graded posets, their definitions can easily be extended to infinite pure simplicial complexes and infinite graded posets. For a discussion of the shellability of infinite simplicial complexes and its topological consequences see [3]. Definition 7.1 applies to infinite as well as finite graded posets. With an obvious extension of the definition of CL-shellability given in [6] to infinite graded posets, the following theorems also hold for infinite as well as finite graded posets.

THEOREM 7.2 [6] *A poset P admits a recursive atom ordering if and only if P is CL-shellable.*

THEOREM 7.3 *If $\Delta({}^JW, S)$ is the order complex of a poset P, then $\hat{P}$ is CL-shellable.*

PROOF. By the argument used in case 2 of Theorem 6.3 and the fact that ordinal sums of CL-shellable posets are CL-shellable ([2]), we can assume that (W, S) is irreducible. By Theorem 6.3, the Coxeter diagram is either linear or $J \supseteq S$. Since the latter case is trivial, we shall assume that S is ordered $s_1, s_2, \ldots, s_n$, so that s_i and s_j commute if $|i-j| \neq 1$. For any $i = 0, 1, \ldots, n-1$, let $r_j = s_{j+i}$, $j \in [n-i]$ and let $R = \{r_1, r_2, \ldots, r_{n-i}\}$. Now let $V = W_R$. For any $w \in {}^JW^R$ let $P_i(w)$ be the poset whose elements are cosets $wuV_{(j)}$, $j = 0, 1, \ldots, n-i+1$, $u \in V$, $wu \in {}^JW$ and with order relation $wuV_{(j)} < wu'V_{(k)}$ if $j > k$ and ${}^JW \cup wuV_{(j)} \cap wu'V_{(k)} \neq \varnothing$. It is not difficult to see that $P_i(w)$ is isomorphic to the interval $[wW_{(i)}, \hat{1}]$ of $\hat{P}$. Hence the order complex of $\overline{P_i(e)}$ is simply $\Delta({}^{J\cap R}V, R)$ and the order complex of $\overline{P_0(e)}$ is $\Delta({}^JW, S)$.

Note that the cosets $wuV_{(1)}$, $u \in V^{(1)}$ are distinct for distinct u and that any atom of $P_i(w)$ is such a coset. If $wuV_{(1)} \in P_i(w)$, where $u \in V^{(1)}$ then by Proposition 4.7 $wu \in {}^JW$. Consequently the atoms of $P_i(w)$ are in one to one correspondence with the elements of $V^{(1)} \cap w^{-1}\,{}^JW$. Hence by ordering the elements of $V^{(1)} \cap w^{-1}\,{}^JW$ we obtain an atom ordering of $P_i(w)$. The theorem is a consequence of the following lemma.

LEMMA 7.4. *Any well ordered extension of Bruhat order on the elements of $V^{(1)} \cap w^{-1}\,{}^JW$ induces a recursive atom ordering of $P_i(w)$.*

PROOF. The proof is by induction on $n-i$ and is trivial for $n-i=1$. Assume $n-i>1$. Let Ω be a well ordering of atoms of $P_i(w)$ such that atom $wu'V_{(1)}$ precedes atom $wuV_{(1)}$ in Ω if $u' < u$ and $u, u' \in V^{(1)}$.

Let $wuV_{(1)}$, $u \in V^{(1)} \cap w^{-1}\,{}^JW$, be any atom of $P_i(w)$. We would like to consider the poset $P_{i+1}(wu)$. But recall that this is defined only for $wu \in {}^JW^{R-\{s_{i+1}\}}$. We already have that $wu \in {}^JW$. If $s \in R - \{s_{i+1}\} = R - \{r_1\}$ then $us > u$ since $u \in V^{(1)}$. Since $w \in W^R$ and $us \in V = W_R$, it follows that $\langle w \rangle \langle u \rangle s$ is reduced. Hence $wus > wu$ from which it follows that $wu \in W^{R-\{s_{i+1}\}}$.

It is not difficult to see that $P_{i+1}(wu)$ is isomorphic to the interval $[wuV_{(1)}, \hat{1}]$ in $P_i(w)$ with isomorphism $\phi(wuvV'_{(k)}) = wuvV_{(k+1)}$ where $V' = W_{R-\{s_{i+1}\}}$. Hence by induction any atom ordering, in which atom $wuv'V_{(2)}$ of $[wuV_{(1)}, \hat{1}]$ precedes atom $wuvV_{(2)}$ for all $v' < v$ and $v', v \in V^{(2)} \cap V_{(1)}$, is a recursive atom ordering. To show that $[wuV_{(1)}, \hat{1}]$ has such an atom ordering which also satisfies (a) of Definition 7.1 it suffices to verify:

PROPERTY 7.5. *For any $wuvV_2$, $wuv'V_{(2)} \in P_i(w)$ such that $v' < v$, and $v', v \in V^{(2)} \cap V_{(1)}$, if $wuvV_{(2)}$ covers an atom $wu'V_{(1)}(u' \in V^{(1)})$ which precedes $wuV_{(1)}$ in Ω then $wuv'V_{(2)}$ must also cover an atom which precedes $wuV_{(1)}$.*

See Figure 2.

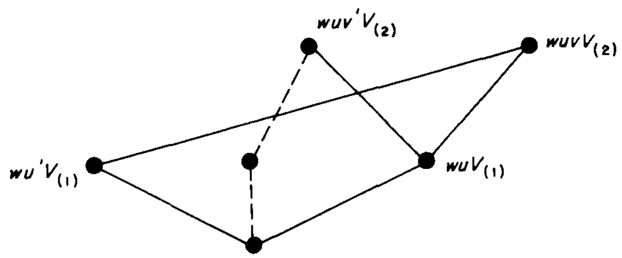


FIGURE 2.

Since $wu'V_{(1)}$ precedes $wuV_{(1)}$ it follows that

$$u \not< u'. \tag{7.1}$$

Since $wuvV_{(2)}$ covers $wu'V_{(1)}$ we have that

$$wuv\alpha = wu'\beta \quad \text{where } \alpha \in V_{(2)}, \beta \in V_{(1)}.$$

If $\alpha \in V_{(1)}$ then $wuV_{(1)} = wuv\alpha V_{(1)} = wu'V_{(1)}$ which is a contradiction. Hence $\alpha \notin V_{(1)}$. Since r_1 commutes with r_i, $i > 2$, it follows that $\alpha = r_1\gamma$, $\gamma \in V_{(1)}$. Hence

$$uvr_1 = u'\beta\gamma^{-1} \quad \text{and} \quad \beta\gamma^{-1} \in V_{(1)}. \tag{7.2}$$

We now show that $uvr_1 < uv$. If not then $u'\beta\gamma^{-1}r_1 < u'\beta\gamma^{-1}$. By the strong exchange property $u'\beta\gamma^{-1}r_1 = \hat{u}'\beta\gamma^{-1}$ where $\hat{u}' < u'$. It follows from (7.2) that $uv = \hat{u}'\beta\gamma^{-1}$ which implies that $uV_{(1)} = \hat{u}'V_{(1)}$. Since $u \in V^{(1)}$, u is the minimum coset representative of $uV_{(1)}$. Hence $u \leq \hat{u}' < u'$ which contradicts (7.1). Consequently

$$uvr_1 < uv. \tag{7.3}$$

Next we show that $uv'r_1 < uv'$. Since $u \in V^{(1)}$ and $v', v \in V_{(1)}$, $\langle u\rangle\langle v\rangle$ and $\langle u\rangle\langle v'\rangle$ are reduced. Hence $v' < v$ implies that $uv' < uv$. If $uv'r_1 > uv'$ then the lifting property (4.3) applied to uvr_1 and uv' yields $uvr_1 \geq uv'$. By the strong exchange property and (7.3) $uvr_1 = \hat{u}v$, where $\hat{u} < u$. By the subword property, $uv' = \hat{\hat{u}}\hat{v}$ where $\hat{v} \leq v$ and $\hat{\hat{u}} \leq \hat{u}$. Consequently $uV_{(1)} = \hat{\hat{u}}V_{(1)}$. This contradicts the minimality of u in $uV_{(1)}$. Hence $uv'r_1 < uv'$ which by the strong exchange property implies that

$$uv'r_1 = \hat{u}v' \quad \text{where} \quad \hat{u} < u. \tag{7.4}$$

It follows that $wuv'r_1 \in w\hat{u}V_{(1)} \cap wuv'V_{(2)}$. To show that $w\hat{u}V_{(1)} < wuv'V_{(2)}$ we verify $wuv'r_1 \in {}^JW$. Since $wuv'V_{(2)} \in P_i(w)$ and $v' \in V^{(2)}$ and $wu \in {}^JW$, by Proposition 4.7 wuv'

$\in {}^JW$. Since $uv'r_1 < uv'$, $uv'r_1$ is a prefix of uv'. Hence again by Proposition 4.7 $wuv'r_1 \in {}^JW$. Since $\hat{u} < u$, the atom $w\hat{u}V_{(1)}$ precedes the atom $wuV_{(1)}$ and Property 7.5 is established.

To verify (b) of Definition 7.1, let $wuV_{(1)}$ and $wu'V_{(1)}$ be atoms of $P_i(w)$ where $u, u' \in V^{(1)}$ and $wu'V_{(1)}$ precedes $wuV_{(1)}$ in Ω. By Proposition 4.7 wu, $wu' \in {}^JW$. Let $wxV_{(j)} \in P_i(w)$ and $wxV_{(j)} > wuV_{(1)}$ and $wxV_{(j)} > wu'V_{(1)}$. Then

$$wxV_{(j)} = wuvV_{(j)} = wu'v'V_{(j)}, \tag{7.5}$$

where v, $v' \in V^{(j)} \cap V_{(1)}$. Again by Proposition 4.7, wuv, $wu'v' \in {}^JW$.

First we show that $uvr_1 < uv$. If $k \neq j$, $vr_k > v$ since $v \in V^{(j)}$. We also have that $vr_k \in V_{(1)}$ if $k \neq 1$. Hence $\langle u\rangle\langle vr_k\rangle$ is reduced if $k \neq 1$, since $u \in V^{(1)}$. Therefore $uvr_k > uv$ if $k \neq 1, j$. If $uvr_1 > uv$ also holds then

$$uv \in V^{(j)}. \tag{7.6}$$

This means that uv is the minimum element of $uvV_{(j)}$ which by (7.5) implies that $uv \leq u'v'$. By the subword property $uv = \hat{u}'\hat{v}'$ where $\hat{u}' \leq u'$ and $\hat{v}' \leq u'$. This implies that $uV_{(1)} = \hat{u}'V_{(1)}$. It follows from $u \in V^{(1)}$ that

$$u \leq \hat{u}' \leq u'. \tag{7.7}$$

Since this contradicts the fact that Ω is induced by an extension of Bruhat order, we have $uvr_1 < uv$.

By the strong exchange property $uvr_1 = \hat{u}v$ where $\hat{u} < u$. By Proposition 4.7 $w\hat{u}v \in {}^JW$. Consequently $w\hat{u}V_{(1)}$ is an atom of $P_i(w)$ which procedes $wuV_{(1)}$ in Ω. We have that

$$w\hat{u}vV_{(2)} > w\hat{u}V_{(1)}$$

and

$$\begin{aligned} w\hat{u}vV_{(2)} &= wuvr_1V_{(2)} \\ &= wuvV_{(2)} \\ &> wuV_{(1)} \end{aligned}$$

and

$$w\hat{u}vV_{(2)} = wuvr_1V_{(2)} \leq wuvV_{(j)}.$$

Hence (b) of Definition 7.1 is established.

THEOREM 7.6. *If $({}^JG, B, N)$ is the order complex of a poset P then $\hat{P}$ is CL-shellable.*

PROOF. Since the proof is very similar to the proof of Theorem 7.3, we omit much of the detail. For $i = 0, 1, \ldots, n-1$ let R and V be as in the proof of Theorem 7.3. For $g \in {}^JG^R$ let $P_i(g)$ be the poset whose elements are cosets $ghBV_{(j)}B$, $j = 0, 1, \ldots, n-i+1$, $h \in BVB$, $gh \in {}^JG$ and with order relation $ghBV_{(j)}B < gh'BV_{(k)}B$ if $j < k$ and ${}^JG \cap ghBV_{(j)}B \cap gh'BV_{(k)}B \neq \varnothing$.

Now the atoms of $P_i(g)$ are of the form $ghBV_{(1)}B$ where $h \in BV^{(1)}B$ and $gh \in {}^JG$. The atoms of $P_i(g)$ are in one to one correspondence with the elements of $(BV^{(1)}B \cap g^{-1}\,{}^JG)/B$. The following lemma completes the proof of the theorem.

LEMMA 7.7. *Let Ω be a well ordering of the atoms of $P_i(g)$ such that if $h' \in Bu'B$, $h \in BuB$, $u, u' \in V^{(1)}$ and $u' < u$ then atom $gh'BV_{(1)}B$ precedes atom $ghBV_{(1)}B$* in Ω. *Then Ω is a recursive atom ordering.*

PROOF. Let $g \in BwB$ and let $ghBV_{(1)}B$ be any atom of $P_i(g)$ where $h \in BuB$ and $u \in V^{(1)}$. Then $gh \in BwuB$ by (3.4). Just as in the proof of Lemma 7.4 $wu \in {}^JW^{R-\{s_{i+1}\}}$ which implies that $gh \in {}^JG^{R-\{s_{i+1}\}}$. Hence the poset $P_{i+1}(gh)$ is defined. It is isomorphic to the interval $[ghBV_{(1)}B, \hat{1}]$ in $P_i(g)$.

We must now establish the property that is analogous to Property 7.5.

PROPERTY 7.8. *For any $ghfBV_{(2)}B, ghf'BV_{(2)}B \in P_i(g)$ such that $f \in BvB$, $f' \in Bv'B$, $v' < v$, and $v', v \in V^{(2)} \cap V_{(1)}$, if $ghfBV_{(2)}B$ covers an atom $gh'BV_{(1)}B$ which precedes $ghBV_{(1)}B$ in Ω* then *$ghf'BV_{(2)}B$ must also cover an atom which precedes $ghBV_{(1)}B$.*

Assume that $h' \in Bu'B$ where $u' \in V^{(1)}$. Since $ghfBV_{(2)}B$ covers $gh'BV_{(1)}B$, it follows that $ghfb_1\alpha b_1' = gh'b_2\beta b_2'$ where $b_1, b_1', b_2, b_2' \in B$ and $\alpha \in V_{(2)}, \beta \in V_{(1)}$. If $\alpha \in V_{(1)}$ then $ghBV_{(1)}B = gh'BV_{(1)}B$ which is impossible. Hence $\alpha \notin V_{(1)}$. It follows that $\alpha = r_1\gamma$, where $\gamma \in V_{(1)} \cap V_{(2)}$. We shall now show that $uvr_1 < uv$. Suppose that $uvr_1 > uv$. Then since $\langle u\rangle\langle v\rangle\langle\gamma\rangle$ is reduced and $uv\gamma r_1 = uvr_1\gamma$, it follows that $\langle u\rangle\langle v\rangle r_1\langle\gamma\rangle$ is reduced. Hence $\langle u\rangle\langle v\rangle\langle\alpha\rangle$ is reduced. This implies that $\langle w\rangle\langle u\rangle\langle v\rangle\langle\alpha\rangle$ is also reduced. By (3.4) and Bruhat decomposition we have that $wuv\alpha = wu'\beta$. We now follow the exact steps of the proof of Lemma 7.4 from (7.2) to obtain (7.3) and (7.4).

By (3.4)

$$ghf' = bwuv'b' \tag{7.8}$$

for some $b, b' \in B$. By (7.4) $bwuv'r_1 \in bw\hat{u}BV_{(1)}B \cap bwuv'BV_{(2)}B$. Just as in the proof of Lemma 7.4, $wuv'r_1 \in {}^JW$. Consequently, $bwuv'r_1 \in {}^JG$. Hence $bw\hat{u}BV_{(1)}B < bwuv'BV_{(2)}B$.

We need only show that $bw\hat{u}BV_{(1)}B$ precedes $ghBV_{(1)}B$ in Ω. We have

$$bwG_R = bwuv'b'G_R = ghf'G_R = gG_R.$$

Since $bw \in G^R$ and $g \in BwB$, $bwB = gB$ by (3.4). Hence $bw\hat{u}BV_{(1)}B = gb''\hat{u}BV_{(1)}B$ where $b'' \in B$. Since $\hat{u} < u$, $gb''\hat{u}BV_{(1)}B$ precedes $ghBV_{(1)}B$ in Ω.

For (b) of Definition 7.1, let $ghBV_{(1)}B$ and $gh'BV_{(1)}B$ be atoms of $P_i(g)$ where $h, h' \in BV^{(1)}B$ and $ghBV_{(1)}B$ precedes $gh'BV_{(1)}B$ in Ω. Let $gxBV_{(j)}B \in P_i(g)$ and $ghBV_{(1)}B$, $gh'BV_{(1)}B < gxBV_{(j)}B$. Then

$$gxBV_{(j)}B = ghfBV_{(j)}B = gh'f'BV_{(j)}B \tag{7.9}$$

where $f, f' \in BV^{(j)}B \cap BV_{(1)}B$. Assume that $g \in BwB$, $h \in BuB$, $h' \in Bu'B$, $f \in BvB$, $f' \in Bv'B$. It follows that $w \in {}^JW^R$, $u, u' \in V^{(1)}$, $v, v' \in V^{(j)} \cap V_{(1)}$. By (3.4) we have $gh \in BwuB$, $gh' \in Bwu'B$, $ghf \in BwuvB$, $gh'f' \in Bwu'v'B$ and

$$hf \in BuvB \quad \text{and} \quad h'f' \in Bu'v'B. \tag{7.10}$$

By Proposition 4.7, $wu, wu', wu'v' \in {}^JW$. Since $gh'BV_{(1)}B$ precedes $ghBV_{(1)}B$ in Ω

$$u \not< u'. \tag{7.11}$$

By (7.9), Bruhat decomposition, and (3.4)

$$wuvV_{(j)} = wu'v'V_{(j)}. \tag{7.12}$$

We now assume $uvr_1 > uv$ and follow the proof of Lemma 7.4 to obtain (7.6) and (7.7). Clearly (7.7) and (7.11) yield $u = u'$. It follows from (7.12) that $vV_{(j)} = v'V_{(j)}$. But $v, v' \in V^{(j)}$ implies that $v = v'$. Hence

$$uv = u'v'. \tag{7.13}$$

By (7.9) we have that $hfBV_{(j)}B = h'f'BV_{(j)}B$. It follows from (7.6), (7.10) and (7.13) that $hf, h'f' \in BV^{(j)}B$. Consequently (3.4) implies that $hfB = h'f'B$. This implies that $ghBV_{(1)}B =$

$ghfBV_{(1)}B = gh'f'BV_{(1)}B = gh'BV_{(1)}B$ which is clearly a contradiction. Hence $uvr_1 < uv$ which by the strong exchange property implies that $uvr_1 = \hat{u}v$ where $\hat{u} < u$. Since $\hat{u}v$ is a prefix of uv, $w\hat{u}v \in {}^JW$. Also since $\hat{u}$ is a prefix of $\hat{u}v$, $w\hat{u} \in {}^JW$.

Since $ghf \in BwuvB$ we let $b \in B$ be such that

$$ghfB = bwuvB.$$

We will show that the coset $bw\hat{u}BV_{(1)}B$ is an atom of $P_i(g)$ that plays the role of a'' in (b) of Definition 7.1. First we must show that

$$gB = bwB \quad \text{and} \quad ghB = bwuB.$$

Clearly $ghBV_{(1)}B = ghfBV_{(1)}B = bwuBV_{(1)}B$. Since $wu \in V^{(1)}$, and $gh \in BwuB$ it follows from (3.4) that $ghB = bwuB$. Similarly $gB = bwB$, which implies that $bw\hat{u}BV_{(1)}B$ is an atom of $P_i(g)$. Clearly $bw\hat{u}BV_{(1)}B$ precedes $ghBV_{(1)}B$ in Ω. We also have that

$$bw\hat{u}vBV_{(2)}B > bw\hat{u}BV_{(1)}B$$

and

$$\begin{aligned} bw\hat{u}vBV_{(2)}B &= bwuvr_1BV_{(2)}B \\ &= bwuvBV_{(2)}B \\ &> bwuBV_{(1)}B \\ &= ghBV_{(1)}B \end{aligned}$$

and

$$\begin{aligned} bw\hat{u}vBV_{(2)}B &= bwuvBV_{(2)}B \\ &\leqslant bwuvBV_{(j)}B \\ &= ghfBV_{(j)}B. \end{aligned}$$

REMARK. The CL-labelings that are induced by the recursive atom orderings in the proofs of Theorems 7.3 and 7.6 in turn induce shelling orders of the maximal chains of the poset. It is not difficult to check that these shellings correspond to linear extensions of Bruhat order on W. Hence they form a subset of the shelling orders obtained by Björner in [3].

8. CONVEX SUBCOMPLEXES

The results of Sections 6 and 7 can be extended to a larger class of subcomplexes. First we need some definitions. A subset U of W is said to satisfy the *prefix property* if for all $w, a, b \in W$ such that b is a prefix of a the following holds: $w, wa \in U$ implies that $wb \in U$. Proposition 4.7 states that ${}^JW$ satisfies the prefix property.

Following Tits [16], we define a *gallery* to be a sequence of chambers such that successive chambers differ by one element. A pure n-dimensional subcomplex Σ of a pure n-dimensional simplicial complex Δ is said to be *convex* if for every pair of chambers C_1 and C_2 in Σ, every gallery of minimal length from C_1 to C_2 is contained in Σ. It is easy to see that if U is a subset of W and $\Delta(U, S)$ is the subcomplex of $\Delta(W, S)$ with chambers C_u, $u \in U$ then $\Delta(U, S)$ is convex if and only if U satisfies the prefix property. It follows that $\Delta({}^JW, S)$ is convex. Since W_J clearly satisfies the prefix property, $\Delta(W_J, S)$ is another convex subcomplex of $\Delta(W, S)$. The following generalizes some results in Sections 6 and 7.

THEOREM 8.1. *Let Σ be a convex subcomplex of $\Delta(W, S)$. If (W, S) has linear connected Coxeter diagram then Σ is the order complex of the proper part of a CL-shellable lattice.*

PROOF. Since Σ is a convex subcomplex of $\Delta(W, S)$, $\Sigma = \Delta(U, S)$ where U is a subset of W with the prefix property. That Σ is the order complex of the proper part of a lattice, follows from the fact that the only property of ${}^JW$ that is used in proving Theorem 6.3 ($\Leftarrow$) and Theorem 6.6 is the prefix property.

In proving Theorem 7.3 all that is used is that ${}^JW$ satisfies the prefix property and that $e \in {}^JW$. One can easily see that if $u \in U$ then the set $u^{-1}U$ satisfies the prefix property. Since $e \in u^{-1}U$, $\Delta(u^{-1}U, S)$ is the proper part of a CL-shellable lattice. The theorem follows from the isomorphism $\Delta(U, S) \simeq \Delta(u^{-1}U, S)$.

Similar arguments can be used to prove a result analogous to Theorem 8.1 for buildings. One can also conclude from Björner's remark 4.15 in [3] and from $\Delta(U, S) \simeq \Delta(u^{-1}U, S)$ that any convex subcomplex of any Coxeter complex or building is shellable. It does not appear that $\Delta({}^JG, B, N)$ is convex. Hence the question of whether the poset with order complex $\Delta({}^JG, B, N)$, is the proper part of a lattice, is open,

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